2015 Australian Intermediate Mathematics Olympiad - Questions

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. A number written in base a is 123_a. The same number written in base b is 146_b. What is the minimum value of $a + b$? [**2 marks**]

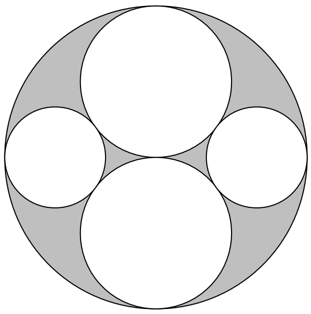
2. A circle is inscribed in a hexagon $ABCDEF$ so that each side of the hexagon is tangent to the circle. Find the perimeter of the hexagon if $AB = 6$, $CD = 7$, and $EF = 8$. [**2 marks**]

3. A selection of 3 whatsits, 7 doovers and 1 thingy cost a total of \$329. A selection of 4 whatsits, 10 doovers and 1 thingy cost a total of \$441. What is the total cost, in dollars, of 1 whatsit, 1 doover and 1 thingy? [**3 marks**]

4. A fraction, expressed in its lowest terms $\frac{a}{b}$, can also be written in the form $\frac{2}{n} + \frac{1}{n^2}$, where n is a positive integer. If $a + b = 1024$, what is the value of a? [**3 marks**]

5. Determine the smallest positive integer y for which there is a positive integer x satisfying the equation $2^{13} + 2^{10} + 2^x = y^2$. [**3 marks**]

6. The large circle has radius $30/\sqrt{\pi}$. Two circles with diameter $30/\sqrt{\pi}$ lie inside the large circle. Two more circles lie inside the large circle so that the five circles touch each other as shown. Find the shaded area.



[**4 marks**]

7. Consider a shortest path along the edges of a 7×7 square grid from its bottom-left vertex to its top-right vertex. How many such paths have no edge above the grid diagonal that joins these vertices? [**4 marks**]

8. Determine the number of non-negative integers x that satisfy the equation

$$\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor.$$

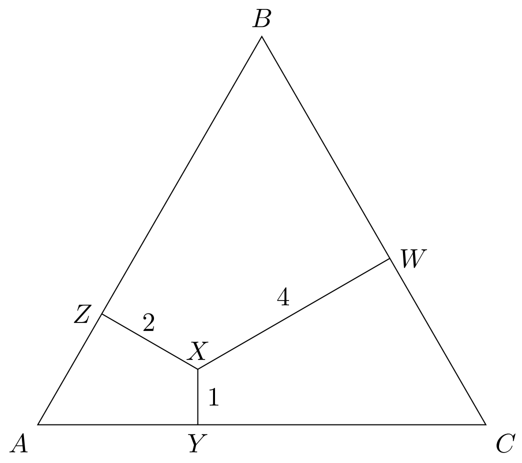
(Note: if r is any real number, then $\lfloor r \rfloor$ denotes the largest integer less than or equal to r.) [**4 marks**]

9. A sequence is formed by the following rules: $s_1 = a, s_2 = b$ and $s_{n+2} = s_{n+1} + (-1)^n s_n$ for all $n \geq 1$.

If $a = 3$ and b is an integer less than 1000, what is the largest value of b for which 2015 is a member of the sequence? Justify your answer. [**5 marks**]

10. X is a point inside an equilateral triangle ABC. Y is the foot of the perpendicular from X to AC, Z is the foot of the perpendicular from X to AB, and W is the foot of the perpendicular from X to BC.

The ratio of the distances of X from the three sides of the triangle is $1 : 2 : 4$ as shown in the diagram.



If the area of $AZXY$ is $13\,\text{cm}^2$, find the area of ABC. Justify your answer. [**5 marks**]

Investigation

If $XY : XZ : XW = a : b : c$, find the ratio of the areas of $AZXY$ and ABC. [**2 bonus marks**]